

A Horn Constraint Solver based on Inductive Theorem Proving

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Background

Verification problems of programs in various paradigms
imperative, logic, concurrent, functional, ...
can be reduced to **Horn constraint solving problems**

Verification Problem

```
let rec mult x y =
  if y=0 then 0 else x + mult x (y - 1)

let rec mult_acc x y a =
  if y=0 then a
  else mult_acc x (y - 1) (a + x)

 $\forall x,y,a. (mult\ x\ y)+a = (mult\_acc\ x\ y\ a)$ 
```

reduce

Horn Constraint Solving Problem

\mathcal{H}_{mult}

$P(x, 0, 0).$
 $P(x, y, x+r) :- P(x, y-1, r) \wedge y \neq 0.$

$Q(x, 0, a, a).$
 $Q(x, y, a, r) :- Q(x, y-1, a+x, r) \wedge y \neq 0.$

Unknown invariant of mult Unknown invariant of mult_acc

$P(x, y, r1) \wedge Q(x, y, a, r2) \models r1+a=r2$
Is there a substitution for P and Q that satisfies all the constraints?

Applications

1. Verif. of higher-order programs

```
type list = Nil | Cons of int * list
let rec sum_list l = match l with
| Nil -> 0 | Cons(x,xs) -> x + sum_list xs
let rec foldl f s l = match l with
| Nil -> s | Cons(x,xs) -> foldl f (f s x) xs
let plus x y = x + y
 $\forall l. sum\_list\ l = foldl\ plus\ 0\ l$ 
```

reduce

$P(Nil, 0).$ $P(Cons(x,l'), x+r) :- P(...).$
 $R(...).$ $R(...)$:- ...
 $P(l, r1) \wedge R(0, l, r2) \models r1=r2$

prove

Spec. Satisfied!

2. Verif. of user-defined ADTs

```
type list =
  Nil | Cons of int * list
let rec append l ys = ...
let rec drop n l = ...
let rec take n l = ...
 $\forall n,l. P(...) \wedge Q(...) \wedge R(...) \models r=1$ 
```

reduce

$P(Nil, l2, l2).$ $P(...)$:- ...
 $Q(...).$ $Q(...)$:- ...
 $R(...).$ $R(...)$:- ...
 $\forall n,l. P(...) \wedge Q(...) \wedge R(...) \models r=1$

prove

Spec. Satisfied!

3. Verif. of programs that may raise exceptions

```
let rec find p l = match l with
| [] -> raise Not_found
| x::xs -> if p x then x else find p xs
let rec find_opt p l = match l with
| [] -> None
| x::xs -> if p x then Some x else
find_opt p xs
 $\forall p,l. Pex(..., Not\_found) \wedge Q(..., r2) \models None=r2$ 
```

reduce

...
 $Pex(..., Not_found) \wedge Q(..., r2) \models None=r2$

prove

Spec. Satisfied!

4. Verif. of heap-manipulating programs

(with Nam Mai and Tachio Terauchi)

```
let rec zero_list x = match !x with
| Nil -> true
| Cons(d,l) -> if d=0 then zero_list l else false
let rec init_list x = match !x with
| Nil -> () | Cons(d,l) -> init_list l; x:=Cons(0,l)
 $\forall x. (init\_list\ x) \Rightarrow (zero\_list\ x)$ 
```

reduce

$Zero_list(h,x) :- h=(x \mapsto Nil) * h2.$
 $Zero_list(h,x) :-$
 $Zero_list(h',l), h=(x \mapsto Cons(0,l)) * h'.$
 $Init_list(h,x,h) :- h=(x \mapsto Nil) * h2.$
 $Init_list(h,x,h') :- Init_list(h2,l,h3),$
 $h=(x \mapsto Cons(d,l)) * h2, h'=(x \mapsto Cons(0,l)) * h3.$
 $\forall h,x,h'. Init_list(h,x,h') \models Zero_list(h',x)$

prove

Spec. Satisfied!

5. Verif. of nonlinear spec.

```
let rec sum n =
  if n<0 then n + sum (n+1)
  else if n=0 then 0
  else n + sum (n-1)
 $\forall n. n \geq 0 \Rightarrow 2 * sum\ n = n * (n+1)$ 
```

reduce

$P(0,0).$ $P(...)$:- $P(...) \wedge x < 0.$
 $p(x, r+x) :- P(x-1, r) \wedge x > 0.$
 $\forall n. P(x, r) \wedge x \geq 0 \models 2 * r = x * (x+1)$

prove

Spec. Satisfied!

Extensions

1. Coinduction on copredicates (with Seiich Shinomiya)

```
merge(Cons(a,as),bs,Cons(a,cs)) :- merge(bs,as,cs).
even(Cons(x1,Cons(x2,xs)),Cons(x1,ys))
:- even(xs,ys).
equal(Cons(l,ls),Cons(r,rs)) :- l=r, equal(ls,rs).
 $\forall u,v,r1,r2. merge(u,v,r1), even(r1,r2) \models equal(r2,u)$ 
```

prove

Solvable!

2. Relational program synthesis

```
let rec mult x y = ...
let rec mult_acc x y a = ?
 $\forall x,y,a. (mult\ x\ y) + a = (mult\_acc\ x\ y\ a)$ 
```

reduce

Horn Constraint Abduction Problem

solve

Definition of mult_acc

Our Horn Constraint Solver (Demo Available!)

Approach : Inductive theorem proving

1. Reduce **Horn constraint solving problems** into **validity checking of first-order formulas with inductively defined predicates**, then
2. Check by **induction on the derivation** of the predicates

Automation of inductive proofs

- (1) A new proof system tailored to Horn constraint solving
- (2) Discharge proof obligations arising in the proof search using
 - PDR-based Horn constraint solver, and
 - SMT solver
- (3) Exploit lemmas, which are
 - User-supplied,
 - Heuristically obtained from the given constraints, or
 - Automatically generated by an abstract interpreter

Advantages

Our method can

- (1) deal with **constraints over any background theories** supported by the underlying SMT solver
nonlinear arithmetics, algebraic data structures, and heaps
- (2) enable verification of **relational specifications** across programs in various paradigms
functional equivalence, associativity, commutativity, monotonicity, ...
- (3) enable verification of recursive functions that are possibly **non-deterministic, higher-order, exception-raising, and over non-inductively defined data types**

Example : Inductive Proof for \mathcal{H}_{mult}

$$\frac{\vdash y \neq 0 \wedge (r_1 - x) + (a + x) = r_2 \Rightarrow r_1 + a = r_2}{\phi; C \wedge y \neq 0 \wedge (r_1 - x) + (a + x) = r_2 \vdash r_1 + a = r_2} \text{Valid}$$

$$\frac{\vdots}{\phi; C \wedge y \neq 0 \vdash r_1 + a = r_2} \text{Apply}$$

$$\frac{\vdots}{\phi; B \wedge y \neq 0 \vdash r_1 + a = r_2} \text{Unfold}$$

$$\frac{\phi; A \vdash r_1 + a = r_2}{A \vdash r_1 + a = r_2} \text{Induct}$$

$\phi = \forall x', y', r_1', r_2'. (P(x', y', r_1') \wedge Q(x', y', a', r_2') \Rightarrow r_1' + a' = r_2')$
 $A = P(x, y, r_1) \wedge Q(x, y, a, r_2)$
 $B = A \wedge P(x, y - 1, r_1 - x)$
 $C = B \wedge Q(x, y - 1, a + x, r_2)$