Formalizing Strong Normalization Proofs

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Strong Normalization Theorem

In typed λ-calculi, strong normalization (SN) theorem is as follows.

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Non-terminating example of a untyped λ-term:

$$\begin{aligned} & (\lambda x.\, x\, x)\, (\lambda x.\, x\, x) \\ \rightarrow_\beta\ & (\lambda x.\, x\, x)\, (\lambda x.\, x\, x) \end{aligned}$$

Our Formalization of the λ-Calculus [Sak15]

- https://github.com/pi8027/lambda-calculus
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  - Formalize many diffrent proofs of the strong normalization theorem in Coq.
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  - untyped λ-calculus
    - Church-Rosser theorem
  - simply typed λ-calculus ($\lambda^{\to}$) and System F ($\lambda 2$)
    - subject reduction theorem
    - strong normalization theorem
      (contains 3 different definitions of the reducibility for each system)

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Part I

Nameless Terms and Proof Automation

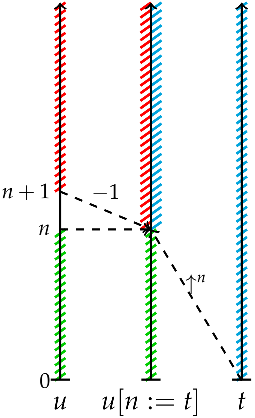
Substitution for Nameless Terms

$$m[n := t] = \begin{cases} m-1 & \text{if } n < m \\ t\uparrow^n & \text{if } n = m \\ m & \text{if } n > m \end{cases}$$

$$(u\,v)[n := t] = u[n := t]\,v[n := t]$$

$$(\lambda u)[n := t] = \lambda u[n+1 := t]$$

$\mathbf{t \uparrow^n}$ is a term which is obtained by adding n to all the free variables of t. This operation is called a **shift** or **lift**.



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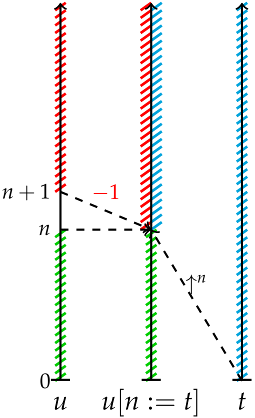

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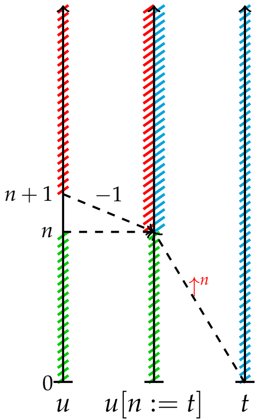

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$\mathbf{t} \uparrow^{\mathbf{d}}_{\mathbf{c}}$ adds d to all the free variable of t that are greater than or equal to c.

$$n \uparrow^d_c = \begin{cases} n + d & \text{if } c \leq n \\ n & \text{if } n < c \end{cases}$$

$$(t\, u) \uparrow^d_c = t \uparrow^d_c \; u \uparrow^d_c$$

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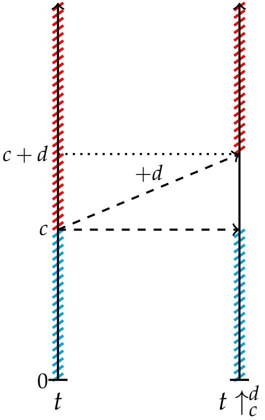
$$t \uparrow^d = t \uparrow^d_0$$



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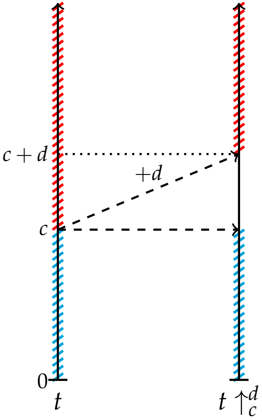

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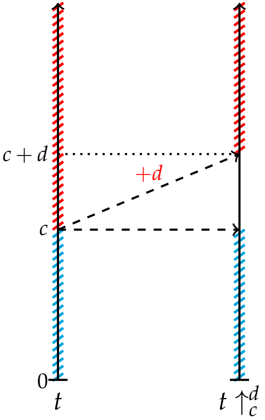

(Restricted) Parallel Substitution

$\mathbf{u}[\mathbf{n} := \mathbf{t_1}, \ldots, \mathbf{t_m}]$ substitutes $t_1, \ldots, t_m$ for free variables $n, \ldots, n+m-1$ in u.

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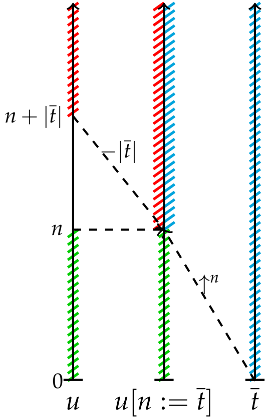

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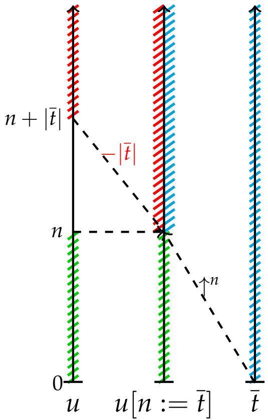

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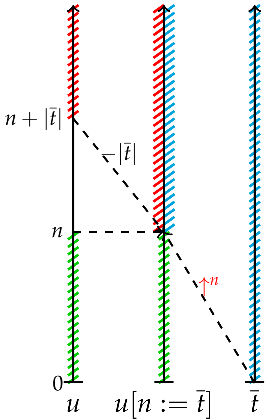

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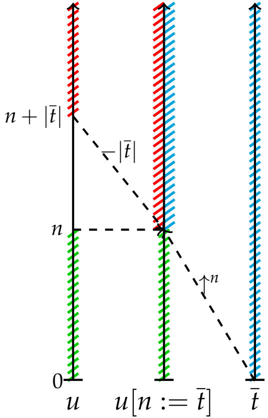
This is useful for proving the strong normalization theorem.



Equational Properties

of Shift and Parallel Substitution

$$t \uparrow^0_n = t \tag{1}$$

$$c \le c' \le c + d \Rightarrow t \uparrow^d_c \uparrow^{d'}_{c'} = t \uparrow^{d'+d}_c \tag{2}$$

$$c' \le c \Rightarrow t \uparrow^d_c \uparrow^{d'}_{c'} = t \uparrow^{d'}_{c'} \uparrow^d_{d'+c} \tag{3}$$

$$c \le n \Rightarrow t[n := \overline{u}] \uparrow^d_c = t \uparrow^d_c [d + n := \overline{u}] \tag{4}$$

$$n \le c \Rightarrow t[n := \overline{u}] \uparrow^d_c = t \uparrow^d_{|\overline{u}|+c} [n := \overline{u} \uparrow^d_{c-n}] \tag{5}$$

$$c \le n \wedge |\overline{u}| + n \le d + c \Rightarrow$$

$$t \uparrow^d_c [n := \overline{u}] = t \uparrow^{d-|\overline{u}|}_c \tag{6}$$

$$m \le n \Rightarrow t[m := \overline{u}][n := \overline{v}] = t[|\overline{u}| + n := \overline{v}][m := \overline{u}[n - m := \overline{v}]] \tag{7}$$

$$t[|\overline{v}| + n := \overline{u}][n := \overline{v}] = t[n := \overline{v} +\!\!+ \overline{u}] \tag{8}$$

$$t[n := []] = t \tag{9}$$

where

$$\bar{t} \uparrow^d_c = [t \uparrow^d_c | \; t \leftarrow \bar{t}]$$

$$\bar{t}[n := \overline{u}] = [t[n := \overline{u}] \mid t \leftarrow \bar{t}]$$

Equational Property (5)

$$n \leq c \Rightarrow t[n := \overline{u}] \uparrow^d_c = t \uparrow^d_{|\overline{u}|+c} [n := \overline{u} \uparrow^d_{c-n}]$$

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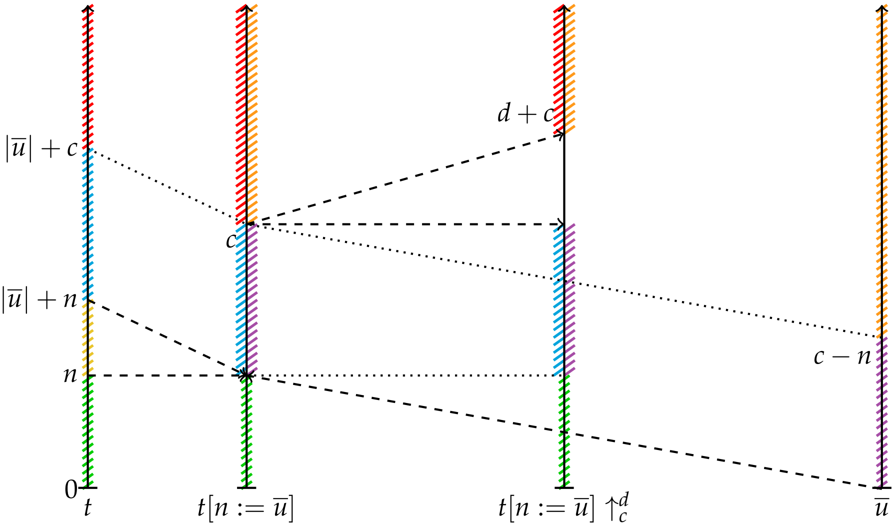

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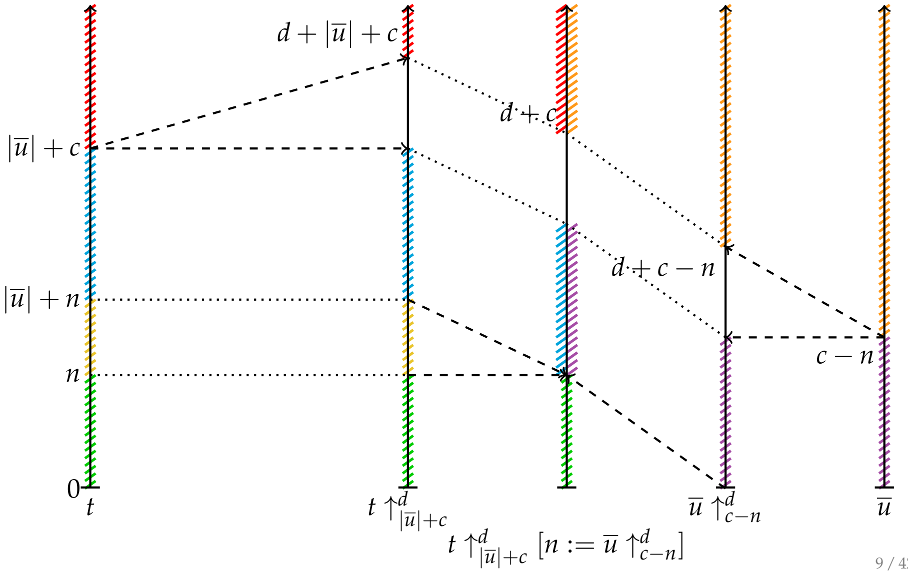

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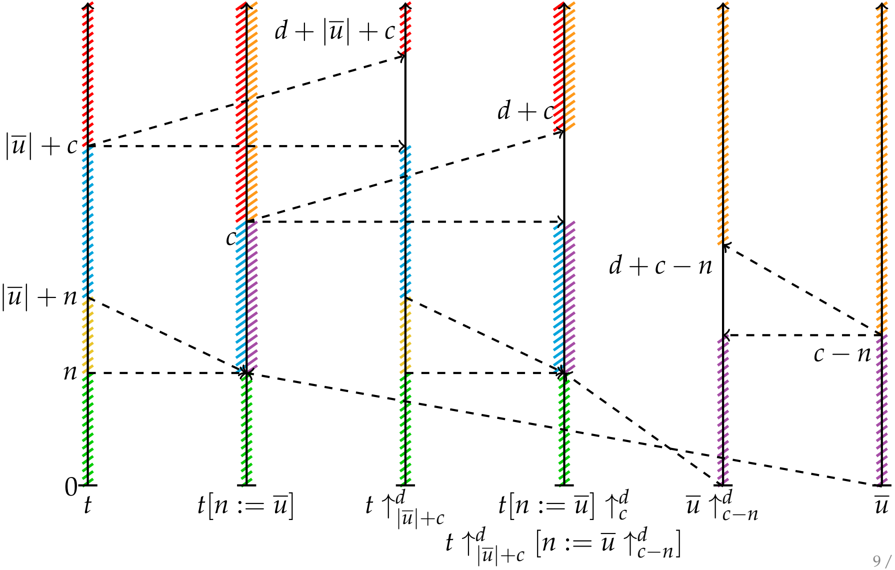

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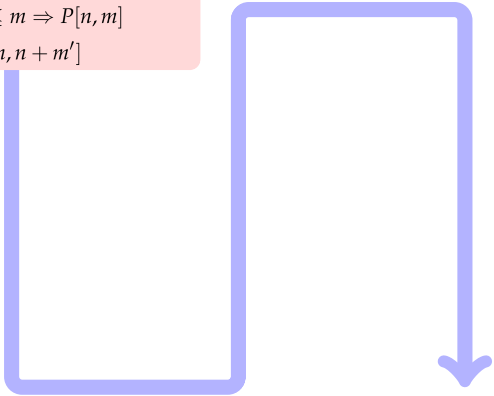

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manual proof

Example

```
Lemma subst_shift_distr n d c ts t :
  n <= c ->
  shift d c (substitute n ts t) =
  substitute n (map (shift d (c - n)) ts)
    (shift d (size ts + c) t).
Proof.
  elimleq; elim: t n; congruence' => v n; elimif_omega.
  - rewrite !nth_default ?size_map /=; elimif_omega.
  - rewrite -shift_shift_distr // nth_map' /=;
      congr shift; apply nth_equal;
      rewrite size_map; elimif_omega.
Qed.
```

Performance Problem of omega

```
Notation minn x y := (x - (x - y)).

Lemma minnA x y z :
  minn x (minn y z) = minn (minn x y) z.
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Some techniques are required to use the `omega` tactic for proving the equational properties.

Part II

Strong Normalization Theorem

$\lambda^{\rightarrow}$: Simply Typed λ-Calculus

types and terms:

$$U ::= X \mid (U \rightarrow U)$$

$$t ::= x \mid (t\,t) \mid (\lambda x : U.\, t)$$

typing rules:

$$\frac{\Gamma(x) = U}{\Gamma \vdash x : U}$$

$$\frac{\Gamma \vdash t : U \rightarrow V \qquad \Gamma \vdash u : U}{\Gamma \vdash t\,u : V}$$

$$\frac{\{x : U\} + \Gamma \vdash t : V}{\Gamma \vdash \lambda x : U.\, t : U \rightarrow V}$$

reduction rules:

$$(\lambda x : U.\, t)\, u \rightarrow_\beta t[x := u]$$

$$\frac{t_1 \rightarrow_\beta t_2}{t_1\, u \rightarrow_\beta t_2\, u} \qquad \frac{u_1 \rightarrow_\beta u_2}{t\, u_1 \rightarrow_\beta t\, u_2}$$

$$\frac{t \rightarrow_\beta t'}{\lambda x : U.\, t \rightarrow_\beta \lambda x : U.\, t'}$$

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$\mathrm{RED}_U(t)$

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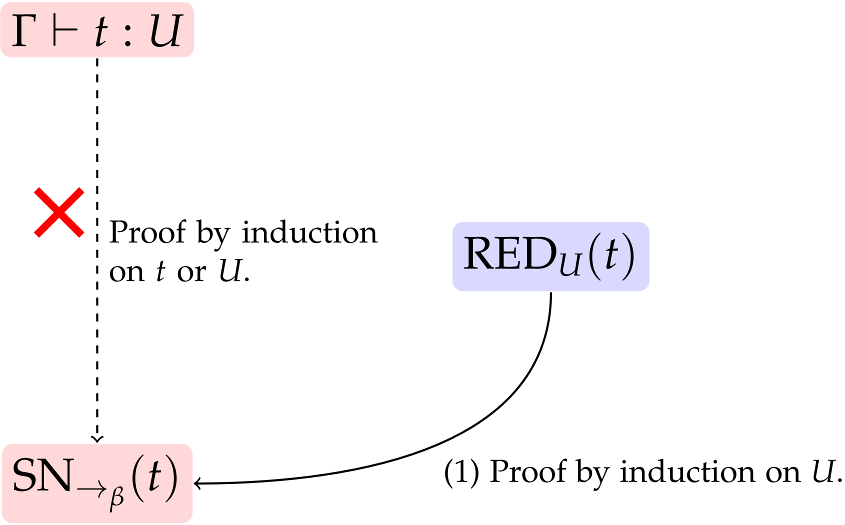

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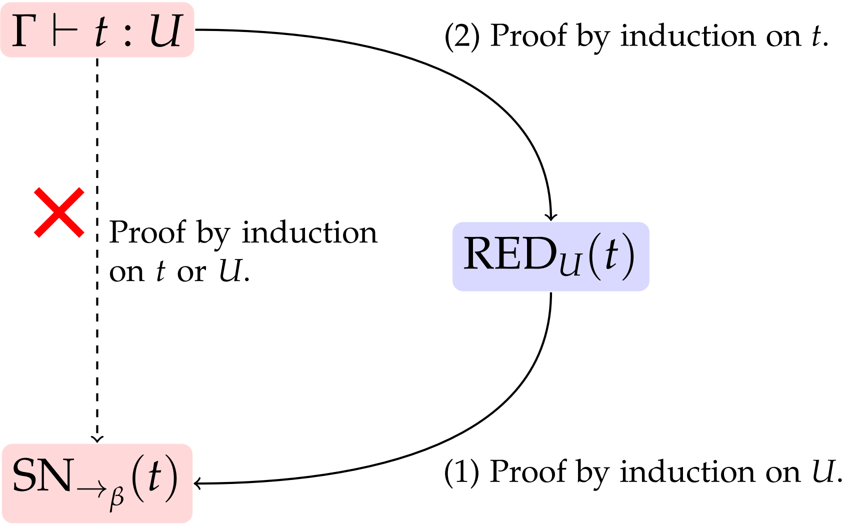

What is RED_U?
in the Simply Typed λ-Calculus

RED_U (**reducibility**) is defined by induction on the type U as follows:

$$\mathrm{RED}_X(t) \stackrel{\text{def}}{\iff} \mathrm{SN}_{\to_\beta}(t)$$
$$\mathrm{RED}_{U\to V}(t) \stackrel{\text{def}}{\iff} \forall u.\, \mathrm{RED}_U(u) \Rightarrow \mathrm{RED}_V(t\,u).$$

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In typical definitions, RED_U is a set of typed terms. But this definition of RED_U contains untyped terms. For example,

$$(\lambda x : U.\, x\,x) \in \mathrm{RED}_X.$$

Part 1: Reducible Terms are SN

CR_1 $\mathrm{RED}_U(t) \Rightarrow \mathrm{SN}_{\to_\beta}(t)$

CR_2 $t \to_\beta t' \wedge \mathrm{RED}_U(t) \Rightarrow \mathrm{RED}_U(t')$

CR_3 $\mathrm{neutral}(t) \wedge (\forall t'.\, t \to_\beta t' \Rightarrow \mathrm{RED}_U(t')) \Rightarrow \mathrm{RED}_U(t)$

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CR_2 is proved by induction on U. $\mathrm{CR}_{1,3}$ are proved together by induction on U.

Part 2: Typed Terms are Reducible

First, we prove the following proposition (reducibility theorem) by induction on t.

$$\begin{aligned}\{x_1 : U_1, \ldots, x_n : U_n, y_1 : V_1, \ldots, y_m : V_m\} \vdash t : U \\ \wedge (\forall i \in \{1, \ldots, n\}.\, \mathrm{RED}_{U_i}(t_i)) \\ \Rightarrow \mathrm{RED}_U(t[x_1, \ldots, x_n := t_1, \ldots, t_n])\end{aligned}$$

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$$\begin{aligned}\{x_1 : U_1, \ldots, x_n : U_n, y_1 : V_1, \ldots, y_m : V_m\} \vdash t : U \\ \wedge (\forall i \in \{1, \ldots, n\}. \, \mathrm{RED}_{U_i}(t_i)) \\ \Rightarrow \mathrm{RED}_U(t[x_1, \ldots, x_n := t_1, \ldots, t_n])\end{aligned}$$

In case of $n = 0$, this proposition is equivalent to

$$\{y_1 : V_1, \ldots, y_m : V_m\} \vdash t : U \Rightarrow \mathrm{RED}_U(t).$$

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$$\{y_1 : V_1, \ldots, y_m : V_m\} \vdash t : U \Rightarrow \mathrm{RED}_U(t).$$

Finally, we get a proof of the strong normalization theorem.

Now, we redefine the reducibility as a set of typed terms.

$$\mathrm{RED'}^{\Gamma}_{X}(t) \stackrel{\text{def}}{\iff} \mathrm{SN}_{\to_\beta}(t)$$
$$\mathrm{RED'}^{\Gamma}_{U\to V}(t) \stackrel{\text{def}}{\iff} \forall u.\, \mathrm{RED}^{\Gamma}_{U}(u) \Rightarrow \mathrm{RED}^{\Gamma}_{V}(t\,u)$$

$$\mathrm{RED}^{\Gamma}_{U}(t) \stackrel{\text{def}}{\iff} \Gamma \vdash t : U \wedge \mathrm{RED'}^{\Gamma}_{U}(t)$$

Typed Reducibility

Unsuccessful Example

Now, we redefine the reducibility as a set of typed terms.

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$$\mathrm{RED'}^{\Gamma}_{U\to V}(t) \overset{\mathrm{def}}{\iff} \forall u.\, \mathrm{RED}^{\Gamma}_{U}(u) \Rightarrow \mathrm{RED}^{\Gamma}_{V}(t\,u)$$

$$\mathrm{RED}^{\Gamma}_{U}(t) \overset{\mathrm{def}}{\iff} \Gamma \vdash t : U \wedge \mathrm{RED'}^{\Gamma}_{U}(t)$$

In this definition, proof of CR_1 is unsuccessful.

A Proof of CR_1
in Untyped Settings

CR_1 $\mathrm{RED}_U(t) \Rightarrow \mathrm{SN}_{\rightarrow_\beta}(t)$
CR_3 $\mathrm{neutral}(t) \wedge (\forall t'.\, t \rightarrow_\beta t' \Rightarrow \mathrm{RED}_U(t')) \Rightarrow \mathrm{RED}_U(t)$

$\mathrm{CR}_{1,3}$ *are proved together by induction on* U.

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Proof. If U is a type variable, CR_1 is a tautology. The only remaining case is $U = V \rightarrow W$.

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Proof. If U is a type variable, CR_1 is a tautology. The only remaining case is $U = V \rightarrow W$.

$$
\begin{aligned}
& \mathrm{RED}_{V\rightarrow W}(t) & \\
\Leftrightarrow\ & \forall u.\, \mathrm{RED}_V(u) \Rightarrow \mathrm{RED}_W(t\,u) & \text{(definition of RED)} \\
\Rightarrow\ & \mathrm{RED}_V(x) \Rightarrow \mathrm{RED}_W(t\,x) & \text{(x is a fresh variable)} \\
\Rightarrow\ & \mathrm{RED}_W(t\,x) & \text{(I.H. of CR}_3\text{)} \\
\Rightarrow\ & \mathrm{SN}_{\rightarrow_\beta}(t\,x) & \text{(I.H. of CR}_1\text{)} \\
\Rightarrow\ & \mathrm{SN}_{\rightarrow_\beta}(t) & \text{(basic property of SN)}
\end{aligned}
$$

A Proof of CR_1
in Untyped Settings

CR_1 $RED_U(t) \Rightarrow SN_{\rightarrow_\beta}(t)$
CR_3 $\text{neutral}(t) \land (\forall t'. t \rightarrow_\beta t' \Rightarrow RED_U(t')) \Rightarrow RED_U(t)$

$CR_{1,3}$ *are proved together by induction on* U*.*

Proof. If U is a type variable, CR_1 is a tautology. The only remaining case is $U = V \rightarrow W$.

$RED_{V \rightarrow W}(t)$	
$\Leftrightarrow \forall u. RED_V(u) \Rightarrow RED_W(t\,u)$	(definition of RED)
$\Rightarrow RED_V(x) \Rightarrow RED_W(t\,x)$	(x is a fresh variable)
$\Rightarrow RED_W(t\,x)$	(I.H. of CR_3)
$\Rightarrow SN_{\rightarrow_\beta}(t\,x)$	(I.H. of CR_1)
$\Rightarrow SN_{\rightarrow_\beta}(t)$	(basic property of SN)

In typed settings, a term of type V is not always existing.

Solutions

There are 2 ways to solve this issue:

- Construct finite set of types by traversing proof tree of $\Gamma \vdash t : U$, and add it to Γ with fresh variables.
- Redefine RED_U as a Kripke logical predicate.

Solution 1: Proof Tree Traversal

$$\mathcal{T}\left(\frac{\Gamma(x) = U}{\Gamma \vdash x : U}\right) = \mathcal{T}'(U)$$

$$\mathcal{T}\left(\frac{P_1 \qquad P_2}{\Gamma \vdash t\, u : V}\right) = \mathcal{T}'(V) \cup \mathcal{T}(P_1) \cup \mathcal{T}(P_2)$$

$$\mathcal{T}\left(\frac{P_1}{\Gamma \vdash \lambda x : U.t : U \to V}\right) = \mathcal{T}'(U \to V) \cup \mathcal{T}(P_1)$$

$$\mathcal{T}'(X) = \emptyset$$

$$\mathcal{T}'(U \to V) = \{U\} \cup \mathcal{T}'(U) \cup \mathcal{T}'(V)$$

Solution 1: Proof Tree Traversal

It is possible to prove the following $\mathrm{CR}_{1,2,3}$ in a similar method.

CR_1 $(\forall V \in \mathcal{T}'(U).\, V \in \Gamma) \wedge \mathrm{RED}^{\Gamma}_{U}(t) \Rightarrow \mathrm{SN}_{\rightarrow_\beta}(t)$

CR_2 $t \rightarrow_\beta t' \wedge \mathrm{RED}^{\Gamma}_{U}(t) \Rightarrow \mathrm{RED}^{\Gamma}_{U}(t')$

CR_3 $(\forall V \in \mathcal{T}'(U).\, V \in \Gamma) \wedge \Gamma \vdash t : U$
$\wedge\, \mathrm{neutral}(t) \wedge (\forall t'.\, t \rightarrow_\beta t' \Rightarrow \mathrm{RED}^{\Gamma}_{U}(t')) \Rightarrow \mathrm{RED}^{\Gamma}_{U}(t)$

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$\wedge\, \text{neutral}(t) \wedge (\forall t'.\, t \rightarrow_\beta t' \Rightarrow \text{RED}^{\Gamma}_{U}(t')) \Rightarrow \text{RED}^{\Gamma}_{U}(t)$

Solution 2: Kripke Logical Predicate

$$
\begin{aligned}
\mathrm{RED}'_X(\Gamma, t) &\stackrel{\text{def}}{\iff} \mathrm{SN}_{\to_\beta}(t) \\
\mathrm{RED}'_{U \to V}(\Gamma, t) &\stackrel{\text{def}}{\iff} \forall \Delta, u.\, \Gamma \lessdot \Delta \land \mathrm{RED}_U(\Delta, u) \Rightarrow \mathrm{RED}_V(\Delta, t\, u) \\
\mathrm{RED}_U(\Gamma, t) &\stackrel{\text{def}}{\iff} \Gamma \vdash t : U \land \mathrm{RED}'_U(\Gamma, t)
\end{aligned}
$$

Solution 2: Kripke Logical Predicate

$\forall x \in \mathrm{dom}(\Gamma).\, \Gamma(x) = \Delta(x)$

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\mathrm{RED}_U(\Gamma, t) &\overset{\text{def}}{\iff} \Gamma \vdash t : U \land \mathrm{RED}'_U(\Gamma, t)
\end{aligned}$$

CR$_1$ $\mathrm{RED}_U(\Gamma, t) \Rightarrow \mathrm{SN}_{\to_\beta}(t)$

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CR$_3$ $\Gamma \vdash t : U \land \mathrm{neutral}(t) \land (\forall t'.\, t \to_\beta t' \Rightarrow \mathrm{RED}_U(\Gamma, t'))$
$\Rightarrow \mathrm{RED}_U(\Gamma, t)$

$\lambda 2$: System F [Gir72, GTL89]

types and terms:

$$U ::= \dots \mid (\Pi X.\, U)$$

$$t ::= \dots \mid (t\, U) \mid (\Lambda X.\, t)$$

additional typing rules:

$$\frac{\Gamma \vdash t : \Pi X.\, U}{\Gamma \vdash t\, V : U[X := V]}$$

$$\frac{X \notin \Gamma \qquad \Gamma \vdash t : U}{\Gamma \vdash \Lambda X.\, t : \Pi X.\, U}$$

additional reduction rules:

$$(\Lambda X.\, t)\, U \to_\beta t[X := U]$$

$$\frac{t_1 \to_\beta t_2}{t_1\, U \to_\beta t_2\, U}$$

$$\frac{t_1 \to_\beta t_2}{\Lambda X.\, t_1 \to_\beta \Lambda X.\, t_2}$$

Strong Normalization Proofs for System F

It is impossible to define a reducibility for System F directly.

Strong Normalization Proofs for System F

It is impossible to define a reducibility for System F directly.
Proof outline of the part 1:

1. Define the **reducibility candidates**. This is a (type indexed) family of terms, and defined by conditions like $\mathrm{CR}_{1,2,3}$.
2. Define the **reducibility with parameters**. This corresponds to the reducibility of $\lambda^{\to}$.
3. Prove that the reducibility with parameters is a reducibility candidate.

Untyped Reducibility Candidates

Set of terms $\mathcal{R}$ is **reducibility candidate** if and only if

CR_1 $\mathcal{R}(t) \Rightarrow \mathrm{SN}(t)$

CR_2 $t \rightarrow_\beta t' \wedge \mathcal{R}(t) \Rightarrow \mathcal{R}(t)$

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↑ t is not of the form $\lambda x.\, u$ or $\Lambda X.\, u$.

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↳ t is not of the form $\lambda x.\, u$ or $\Lambda X.\, u$.

For example, SN is a reducibility candidate.

Untyped Reducibility with Parameters

$$\mathrm{RED}_Y[\overline{X} := \overline{\mathcal{R}}](t) \overset{\text{def}}{\Longleftrightarrow} \begin{cases} \overline{\mathcal{R}}_i(t) & \text{if } \; Y = \overline{X}_i \\ \mathrm{SN}(t) & \text{if } \; Y \notin \overline{X} \end{cases}$$

$$\mathrm{RED}_{U \to V}[\overline{X} := \overline{\mathcal{R}}](t) \overset{\text{def}}{\Longleftrightarrow} \forall u.\, \mathrm{RED}_U[\overline{X} := \overline{\mathcal{R}}](u) \Rightarrow \mathrm{RED}_V[\overline{X} := \overline{\mathcal{R}}](t\, u)$$

$$\mathrm{RED}_{\Pi Y.\, U}[\overline{X} := \overline{\mathcal{R}}](t) \overset{\text{def}}{\Longleftrightarrow} \forall V, \mathcal{S}.\, \mathrm{RC}(\mathcal{S}) \Rightarrow \mathrm{RED}_U[Y, \overline{X} := \mathcal{S}, \overline{\mathcal{R}}](t\, V)$$

Untyped Reducibility with Parameters

$$\mathrm{RED}_Y[\overline{X} := \overline{\mathcal{R}}](t) \overset{\text{def}}{\iff} \begin{cases} \overline{\mathcal{R}}_i(t) & \text{if } Y = \overline{X}_i \\ \mathrm{SN}(t) & \text{if } Y \notin \overline{X} \end{cases}$$

$$\mathrm{RED}_{U \to V}[\overline{X} := \overline{\mathcal{R}}](t) \overset{\text{def}}{\iff} \forall u.\, \mathrm{RED}_U[\overline{X} := \overline{\mathcal{R}}](u) \Rightarrow \mathrm{RED}_V[\overline{X} := \overline{\mathcal{R}}](t\,u)$$

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Lemma If $\overline{\mathcal{R}}$ is a sequence of reducibility candidates, $\mathrm{RED}_U[\overline{X} := \overline{\mathcal{R}}]$ is a reducibility candidate.

Typed Reducibility Candidates 1 [Hur10]

Set of terms $\mathcal{R}$ is reducibility candidate of Γ, U if and only if

CR$_1$ $\#\Gamma \vdash t : U \wedge \mathcal{R}(t) \Rightarrow \mathrm{SN}(t)$

CR$_2$ $\#\Gamma \vdash t : U \wedge t \rightarrow_\beta t' \wedge \mathcal{R}(t) \Rightarrow \mathcal{R}(t')$

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where $\#\Gamma = \{b : \Pi X.\, X\} + \Gamma$

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where $\#\Gamma = \{b : \Pi X.\, X\} + \Gamma$

↑

$\#\Gamma \vdash b\,U : U$

$b\,U$ is neutral and normal

Typed Reducibility with Parameters 1

$$\mathrm{RED}^{\Gamma}_{Y}[\overline{X} := \overline{\mathcal{R}} : \overline{U}](t) \overset{\text{def}}{\iff} \begin{cases} \overline{\mathcal{R}}_i(t) & \text{if } \ Y = \overline{X}_i \\ \mathrm{SN}(t) & \text{if } \ Y \notin \overline{X} \end{cases}$$

$$\mathrm{RED}^{\Gamma}_{V \to W}[\overline{X} := \overline{\mathcal{R}} : \overline{U}](t) \overset{\text{def}}{\iff} \forall u.\, \#\Gamma \vdash u : V[\overline{X} := \overline{U}] \Rightarrow \mathrm{RED}^{\Gamma}_{V}[\overline{X} := \overline{\mathcal{R}} : \overline{U}](u) \Rightarrow \mathrm{RED}^{\Gamma}_{W}[\overline{X} := \overline{\mathcal{R}} : \overline{U}](t\,u)$$

$$\mathrm{RED}^{\Gamma}_{\Pi Y.\,V}[\overline{X} := \overline{\mathcal{R}} : \overline{U}](t) \overset{\text{def}}{\iff} \forall W, \mathcal{S}.\, \mathrm{RC}^{\Gamma}_{W}(\mathcal{S}) \Rightarrow \mathrm{RED}^{\Gamma}_{V}[Y, \overline{X} := \mathcal{S}, \overline{\mathcal{R}} : W, \overline{U}](t\,W)$$

Typed Reducibility with Parameters 1

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$$\mathrm{RED}^{\Gamma}_{\Pi Y.\,V}[\overline{X} := \overline{\mathcal{R}} : \overline{U}](t) \stackrel{\text{def}}{\iff} \forall W, \mathcal{S}.\, \mathrm{RC}^{\Gamma}_{W}(\mathcal{S}) \Rightarrow \mathrm{RED}^{\Gamma}_{V}[Y, \overline{X} := \mathcal{S}, \overline{\mathcal{R}} : W, \overline{U}](t\,W)$$

Lemma If $\overline{\mathcal{R}}_i$ is a reducibility candidate of $\Gamma, \overline{U}_i$ for all $i \leq |\overline{X}|$, $\mathrm{RED}^{\Gamma}_{V}[\overline{X} := \overline{\mathcal{R}} : \overline{U}]$ is a reducibility candidate of $\Gamma, V[\overline{X} := \overline{U}]$.

Typed Reducibility Candidates 2 [Gal89]

Set of pairs of type environment and term $\mathcal{R}$ is reducibility candidate of type U if and only if

CR$_{\text{typed}}$ $\mathcal{R}(\Gamma, t) \Rightarrow \Gamma \vdash t : U$

CR$_0$ $\Gamma \leq \Delta \wedge \mathcal{R}(\Gamma, t) \Rightarrow \mathcal{R}(\Delta, t)$

CR$_1$ $\mathcal{R}(t) \Rightarrow \mathrm{SN}(t)$

CR$_2$ $t \rightarrow_\beta t' \wedge \mathcal{R}(t) \Rightarrow \mathcal{R}(t')$

CR$_3$ $\Gamma \vdash t : U \wedge \mathrm{neutral}(t) \wedge (\forall t'.t \rightarrow_\beta t' \Rightarrow \mathcal{R}(t')) \Rightarrow \mathcal{R}(t)$.

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Typed Reducibility with Parameters 2

$$\mathrm{RED}_Y[\overline{X} := \overline{\mathcal{R}} : \overline{U}](\Gamma, t) \overset{\text{def}}{\iff} \begin{cases} \overline{\mathcal{R}}_i(\Gamma, t) & \text{if } Y = \overline{X}_i \\ \mathrm{SN}'(\Gamma, t) & \text{if } Y \notin \overline{X} \end{cases}$$

$$\begin{aligned} \mathrm{RED}_{V \to W}[\overline{X} := \overline{\mathcal{R}} : \overline{U}](\Gamma, t) \overset{\text{def}}{\iff} \; & \Gamma \vdash t : V \to W \\ & \wedge (\forall \Delta, u.\, \Gamma \leq \Delta \\ & \qquad \Rightarrow \mathrm{RED}_V[\overline{X} := \overline{\mathcal{R}} : \overline{U}](\Delta, u) \\ & \qquad \Rightarrow \mathrm{RED}_W[\overline{X} := \overline{\mathcal{R}} : \overline{U}](\Delta, t\, u)) \end{aligned}$$

$$\begin{aligned} \mathrm{RED}_{\Pi Y.\, V}[\overline{X} := \overline{\mathcal{R}} : \overline{U}](\Gamma, t) \overset{\text{def}}{\iff} \; & \forall W, \mathcal{S}.\, \mathrm{RC}_W(\mathcal{S}) \\ & \qquad \Rightarrow \mathrm{RED}_V[Y, \overline{X} := \mathcal{S}, \overline{\mathcal{R}} : W, \overline{U}](t\, W) \end{aligned}$$

Typed Reducibility with Parameters 2

$$\mathrm{RED}_Y[\overline{X} := \overline{\mathcal{R}} : \overline{U}](\Gamma, t) \overset{\text{def}}{\iff} \begin{cases} \overline{\mathcal{R}}_i(\Gamma, t) & \text{if } \ Y = \overline{X}_i \\ \mathrm{SN}'(\Gamma, t) & \text{if } \ Y \notin \overline{X} \end{cases}$$

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Lemma If $\overline{\mathcal{R}_i}$ is a reducibility candidate of $\overline{U}_i$ for all $i \leq |\overline{X}|$, $\mathrm{RED}_V[\overline{X} := \overline{\mathcal{R}} : \overline{U}]$ is a reducibility candidate of $V[\overline{X} := \overline{U}]$.

Comparison of the SN Proofs

- SN proofs with typed reducibility requires type preservation lemmas. On the other hand, SN proofs with untyped reducibility are completed without type preservation lemmas. (Untyped proofs are relatively simple.)
- Typed reducibilities are capturing the features of reducible terms.

Conclusion

- We formalized strong normalization proofs with 6 different definitions of the reducibility.
- `$ wc -lc **/*.v`

  ```
  ...
    1808  72327 coq/LC/Debruijn/F.v
     647  24413 coq/LC/Debruijn/STLC.v
  ...
    3746 138149 total
  ```
- https://github.com/pi8027/lambda-calculus

Appendix

λ-Calculus and Representations of Binding

named representation
(name-carrying term)

$$
\begin{aligned}
t ::= {} & x \ (\in \mathbf{Var}) \\
\mid {} & (t\, t) \\
\mid {} & (\lambda x.\, t)
\end{aligned}
$$

de Bruijn representation [dB72]
(nameless terms)

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\begin{aligned}
t ::= {} & x \ (\in \mathbb{N}) \\
\mid {} & (t\, t) \\
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λ-Calculus and Representations of Binding

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\end{aligned}
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de Bruijn representation [dB72] (nameless terms)

$$
\begin{aligned}
t ::= &\ x \ (\in \mathbb{N}) \longleftarrow \\
\mid &\ (t\, t) \\
\mid &\ (\lambda t)
\end{aligned}
$$

This number indicates the index of corresponding binder.

λ-Calculus and Representations of Binding

named representation (name-carrying term)

$$t ::= x \ (\in \mathbf{Var}) \mid (t\, t) \mid (\lambda x.\, t)$$

de Bruijn representation [dB72] (nameless terms)

$$t ::= x \ (\in \mathbb{N}) \mid (t\, t) \mid (\lambda t)$$

← This number indicates the index of corresponding binder.

↑ Nameless terms don't require a variable name in binding positions.

λ-Calculus and Representations of Binding

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de Bruijn representation [dB72] (nameless terms)

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This number indicates the index of corresponding binder.

Nameless terms don't require a variable name in binding positions.

examples:

$\lambda x.\,\lambda y.\,(\lambda z.\,y\,x\,z)\,x\,a$

$\lambda\lambda(\lambda 1\,2\,0)\,1\,2$

λ-Calculus and Representations of Binding

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λ-Calculus and Representations of Binding

named representation (name-carrying term)

$$t ::= x \;\; (\in \mathbf{Var}) \;\mid\; (t\,t) \;\mid\; (\lambda x.\, t)$$

de Bruijn representation [dB72] (nameless terms)

$$t ::= x \;\; (\in \mathbb{N}) \;\mid\; (t\,t) \;\mid\; (\lambda t)$$

This number indicates the index of corresponding binder.

Nameless terms don't require a variable name in binding positions.

examples:

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\begin{aligned}
t ::= {} & x \quad (\in \mathbf{Var}) \\
\mid {} & (t\,t) \\
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We use the de Bruijn representation for our formalization.

(Informal) Definition of Reduction

$$(\lambda x.\, t)\, u \rightarrow_\beta t[x := u]$$

$$\frac{t_1 \rightarrow_\beta t_2}{t_1\, u \rightarrow_\beta t_2\, u} \qquad \frac{u_1 \rightarrow_\beta u_2}{t\, u_1 \rightarrow_\beta t\, u_2} \qquad \frac{t \rightarrow_\beta t'}{\lambda x.\, t \rightarrow_\beta \lambda x.\, t'}$$

(Informal) Definition of Reduction

substituting u for every free occurrence of x in t

$$(\lambda x.\, t)\, u \to_\beta t[x := u]$$

$$\frac{t_1 \to_\beta t_2}{t_1\, u \to_\beta t_2\, u} \qquad \frac{u_1 \to_\beta u_2}{t\, u_1 \to_\beta t\, u_2} \qquad \frac{t \to_\beta t'}{\lambda x.\, t \to_\beta \lambda x.\, t'}$$

Capture

$$(\lambda y.\, \lambda x.\, y)\, x$$

Capture

free occurrence of x

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free occurrence of x

$$(\lambda y.\, \lambda x.\, y)\; x$$

$$\to_\beta (\lambda x.\, y)[y := x]$$

Capture

free occurrence of x

$$
\begin{aligned}
&(\lambda y.\, \lambda x.\, y)\, x \\
\rightarrow_\beta\ &(\lambda x.\, y)[y := x] \\
=\ &\lambda x.\, x
\end{aligned}
$$

Capture

free occurrence of x

$$(\lambda y.\, \lambda x.\, y)\; x$$

$$\rightarrow_\beta (\lambda x.\, y)[y := x]$$

$$= \lambda x.\, x$$

x is bound variable. (captured)

Capture

free occurrence of x

$$(\lambda y.\, \lambda x.\, y)\, x$$

$$\rightarrow_\beta (\lambda x.\, y)[y := x]$$

$$= \lambda x.\, x$$

x is bound variable. (captured)

In the named representation, it is necessary to use a restricted reduction rule and the α-equivalence relation or capture-avoiding substitutions.

Comparison of the Representations

- named representation
  - Non essential part of the proofs relevant to bindings are large.
  - Most part of proofs are required conditions relevant to free variables such as $x \notin \mathrm{FV}(t)$, $\mathrm{FV}(t) \cap \mathrm{FV}(t') = \emptyset$, etc.
- de Bruijn representation
  - We can concentrate on the essential part of the proofs.
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  - We can concentrate on the essential part of the proofs.
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- The set of nameless terms corresponds to the quotient set of the named terms by α-equivalence relation.

Strong Normalization Property

The set of strongly normalizable terms $\mathrm{SN}_{\rightsquigarrow} \subseteq A$ can be defined by following axioms.

SN-Intro $\forall x \in A. (\forall y \in A. x \rightsquigarrow y \Rightarrow \mathrm{SN}_{\rightsquigarrow}(y)) \Rightarrow \mathrm{SN}_{\rightsquigarrow}(x)$

SN-Elim $\forall P \subseteq A. (\forall x \in A. (\forall y \in A. x \rightsquigarrow y \Rightarrow \mathrm{SN}_{\rightsquigarrow}(y) \wedge P(y)) \Rightarrow P(x)) \Rightarrow \mathrm{SN}_{\rightsquigarrow} \subseteq P$

Strong Normalization Property

The set of strongly normalizable terms $\mathrm{SN}_{\leadsto} \subseteq A$ can be defined by following axioms.

SN-Intro	$\forall x \in A.\,(\forall y \in A.\, x \leadsto y \Rightarrow \mathrm{SN}_{\leadsto}(y)) \Rightarrow \mathrm{SN}_{\leadsto}(x)$
SN-Elim	$\forall P \subseteq A.\,(\forall x \in A.\,(\forall y \in A.\, x \leadsto y \Rightarrow \mathrm{SN}_{\leadsto}(y) \wedge P(y)) \Rightarrow P(x)) \Rightarrow \mathrm{SN}_{\leadsto} \subseteq P$

In the Coq standard library, the strong normalization property is defined as a inductive predicate `Acc`. Constructor and induction principle of `Acc` correspond to SN-Intro and SN-Elim.

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